Universidad Nacional de La Plata

Departamento de Economía
Facultad de Ciencias Económicas
Universidad Nacional de La Plata

A Geometric Representation of the Frisch-Waugh-Lovell Theorem

Walter Sosa Escudero

Documento de Trabajo Nro. 29
Marzo 2001

A Geometric Representation of the Frisch-Waugh-Lovell Theorem*

Walter Sosa Escudero†
Universidad Nacional de La Plata

March 21, 2001

1 Introduction

Even though the result recently referred to as the 'Frisch-Waugh-Lovell theorem' (FWL theorem, henceforth) has been around for a long time, it is relatively recently that it has been widely used by econometricians as a powerful pedagogical tool to express in a simple and intuitive way many results that often rely on tedious and seldom intuitive algebraic steps, which are also notationally cumbersome.

Even though a proof of the FWL theorem can be based entirely on standard algebraic results, the main reason of its increasing popularity is its strong *geometric* appeal. Recent texts and articles provide a mix between algebraic proofs and geometrical illustrations of the theorem, but none of them presents a fully geometrical proof of the result. The goal of this note is very modest: it extends the standard geometrical representations of the theorem to actually prove it based on geometrical arguments, which should, hopefully, provide a richer understanding of the scope of the theorem.

2 The Frisch-Waugh-Lovell Theorem

This note can be seen as an addendum to the presentation in recent texts in advanced econometrics like Davidson and MacKinnon (1993) or Ruud (2000), which provide extensive coverage of the theorem. For simplicity, we will follow the former. The setup of the theorem is the standard linear model in matrix form:

*I thank Alvaro Mezza for a careful reading of a previous version, and the participants of the Econometrics I Seminar, where this note originated. The usual disclaimer applies.
†Departamento de Economia, Facultad de Ciencias Economicas, Universidad Nacional de La Plata, calle 6 entre 47 y 48, oficina 516, 1900 La Plata, Argentina. Phone-fax: 54-221-4229383. email: wsosa@feedback.net.ar

$$Y = X\beta + u$$

where Y is an n vector of observations of the dependent variable, X is a $n \times k$ non-stochastic matrix of observations of k explanatory variables, and u is a vector of error terms. Let's partition X so the model is expressed as follows:

$$Y = X_1\beta_1 + X_2\beta_2 + u \tag{1}$$

where X_1 and X_2 are matrices of observations of k_1 and k_2 explanatory variables, and β_1 and β_2 are the corresponding coefficients vectors. Consequently, $X = [X_1 \;\; X_2]$, $\beta' = (\beta_1' \;\; \beta_2')'$ and $k = k_1 + k_2$.

Let $M_1 = I - X_1(X_1'X_1)^{-1}X_1'$, that is, M_1 is an orthogonal projection matrix that projects any vector in R^n onto the orthogonal complement of the linear space spanned by the columns of X_1. Let $Y^* = M_1Y$ and $X_2^* = M_1X_2$. Y^* and X_2^* are, respectively, OLS residuals of regressing Y and all the columns of X_2 on X_1.

Suppose that we are interested in estimating β_2 in (1), and consider the following alternative methods:

- *Method 1*: Proceed as usual and regress Y on X obtaining the OLS estimator $\hat{\beta} = (\hat{\beta}_1' \;\; \hat{\beta}_2')' = (X'X)^{-1}X'Y$. $\hat{\beta}_2$ would be the desired estimate.

- *Method 2*: Regress Y^* on X_2^* and obtain as estimate $\tilde{\beta}_2 = (X_2^{*\prime}X_2^*)^{-1}X_2^{*\prime}\,Y^*$

Let e_1 and e_2 be the residuals vectors of the regressions in Method 1 and 2, respectively. Now we can state the theorem.

Theorem (Frisch and Waugh, 1933, Lovell, 1963): *$\hat{\beta}_2 = \tilde{\beta}_2$ (first part) and $e_1 = e_2$ (second part).*

The theorem says that both methods yield exactly the same estimates of β_2 and that residuals of both regressions are the same. That is, an estimate of β_2 can be obtained by directly regressing Y on X_1 and X_2 or in a two-step fashion. In the first step, we 'get rid' of the effect of X_1 by substracting to Y and X_2 the part of them that can be linearly explained by X_1, and in the second part we run a simple regression using this 'cleaned' variables (Y^* and X_2^*).

Technically, Method 1 projects Y on the space spanned by the columns of X, and its residuals are projections of Y on the orthogonal complement of such space. Method 2 decomposes this procedure in two steps. The first step 'eliminates' the effect of X_1 by first projecting Y and X_2 on the orthogonal complement of the space spanned by the columns of X_1, that is, it creates new variables Y^* and X_2^* which are OLS residuals of regressing Y and X_2 on X_1. The second step simply runs OLS on these transformed variables, that is, Y^* is projected orthogonally on the space spanned by X_2^*, which, by construction, is orthogonal to the space spanned by X_1.

Simple as it looks, the FWW theorem is a very powerful tool to understand the mechanics of OLS estimation. Even though there are several algebraic ways to prove the theorem (one of them is presented in the Appendix) a geometrical representation helps notoriusly to understand how the OLS method works.

3 A geometrical representation

The geometrical representation presented in this note extends that in Davidson and MacKinnon (1993, pp. 22). For simplicity, let us consider the case where $k_1 = k_2 = 1$, that is, there are only two explanatory variables[1]. Figure 1 shows the three main vectors involved in the OLS estimation of (1). Y, X_1 and X_2 are vectors in a three-dimensional euclidean vector space. Data vectors are represented with arrows and labeled with bold letters. Lowercase letters represent points. OLS projects Y on the space spanned by X_1 and X_2, which, in this case has dimension two. The OLS projection is represented by the vector $ob = PY$ where $P = X(X'X)^{-1}X'$ is the matrix that projects Y orthogonally on the span of X. The residual vector is $ab = MY$ where $M = I - P$ is the matrix that projects Y on the orthogonal complement of the span of X. Given that $PY = X_1\hat{\beta}_1 + X_2\hat{\beta}_2$, the coordinates of vectors $oe = X_1\hat{\beta}_1$ and $od = X_2\hat{\beta}_2$ can be easily found using the parallelogram's law. This provides the geometrical representation of all the elements involved in Method 1.

[INSERT FIGURE 1 HERE]

In order to explore the geometry of the second method, first let us project Y orthogonally on the span of X_1, which is represented by $oc = P_1Y$ and the corresponding residual vector $ac = Y^* = M_1Y$. Now do the same with X_2. The projection of X_2 on X_1 is represented by $og = P_1X_2$ and the residuals vector is $fg = X_2^* = M_1X_2$. The second methods regresses Y^* on the span of X_2^*, which is represented by the line containing segment cg, which is simply fg translated so as it has origin in c. This projection gives the vector cb and the corresponding residuals vector is, trivially, the vector ab. This illustrates the second part of the theorem: OLS residuals of both methods are exactly the same.

[INSERT FIGURE 2 HERE]

[1] Perhaps one of the most interesting corollaries of the FTW theorem is that one can reduce all the relevant aspects of the multivariable case to the two variable case.

Even though the first part of the theorem can also be easily explored in the same picture, in order to avoid cluttering Figure 1 too much, let's look at Figure 2, which is simply Figure 1 seen 'from above'. From Method 1, $X_2\hat{\beta}_2 = od = of\hat{\beta}_2$, and from Method 2, $X_2\tilde{\beta}_2 = cb = cj\tilde{\beta}_2$. Now by Thales' Theorem $od/of = cb/cj$, and replacing we get the first part of the theorem: $\hat{\beta}_2 = \tilde{\beta}_2$

4 Historic coda

The result behind the FWL theorem has been known in the econometrics literature for a long time. In fact, if everything contained in the first volume of *Econometrica* can be regarded as 'seminal' of 'foundational', this is surely the case of the FWL theorem. Moreover, almost every intermediate to advanced econometrics text refer to it either explicitely or indirectly, exploiting its geometrical structure in varying degrees. Davidson and MacKinnon's (1993) text labelled it as the 'Frisch-Waugh-Lovell' Theorem in honour of the Frisch and Waugh (1933) paper where the result is proved for the first time in econometrics, and the paper by Lovell (1963), which presents a nice application of the result. The name seems to attempt to do justice with the originality of the result (Frisch and Waugh) and its applicability (Lovell), much more ambitious than a casual look would suggest. Davidson and MacKinnon devote an entire chapter (and many subsequent references) in their book to the FWL theorem. The WWW based 'Dictionary of Economics' has an entry labelled 'Frisch-Waugh-Lovell Theorem'. In spite of DM's effort in giving credit to the three authors, there is still no agreement in the profession regarding how to refer to these results. For example, Fiebig, et al. call it the 'Frisch-Waugh' theorem, dropping Lovell's name, and a very recent text by Paul Ruud (2000), though devoting extensive coverage to the subject (even more than Davidson and MacKinnon), still refer to it as the 'partitioned regression' theorem. Goldberger's classic text also gives detailed treatment, refering to it as the 'residual regression' approach.

References

Davidson, R. and MacKinnon, R., 1993, *Estimation and Inference in Econometrics*, Oxford University Press, Oxford.

Fiebig, D., Bartels, R. and Kramer, W., 1996, The Frisch-Waugh Theorem and Generalized Least Squares, *Econometric Reviews*, 15(4), pp. 431-443.

Frisch, R. and Waugh, F., 1933, Partial time regressions as compared with individual trends, *Econometrica*, 45, 939-53.

Goldberger, A., 1991, *A Course in Econometrics*, Harvard University Press, Cambridge.

Lovell, M., 1963, Seasonal adjustment of econmic time series, *Journal of the American Statistical Association*, 58, 993-1010.

Ruud, P., 2000, *An Introduction to Classical Econometric Theory*, Oxford University Press, Oxford.

Appendix: Algebraic proof of the theorem

For completeness, we give a standard algebraic proof of the theorem. The starting point is the orthogonal decomposition:

$$Y = PY + MY = X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + MY$$

To prove the first part, multiply both sides by $X_2'M_1$ and get:

$$X_2'M_1Y = X_2'M_1X_1\hat{\beta}_1 + X_2'M_1X_2\hat{\beta}_2 + X_2'M_1MY$$

The first term of the right hand side vanishes since, by definition, M_1 projects X_1 on its orthogonal complement, so $M_1X_1 = 0$. The third term vanishes too since $X_2'M_1M = X_2'M - P_1X_2'M$ and $X_2'M = 0$ for the same reasons as before. Then, we are left only with the second term. Solving for $\hat{\beta}_2$ proves the first part of the theorem.

To prove the second part multiply the orthogonal decomposition by M_1 and obtain:

$$M_1Y = M_1X_1\hat{\beta}_1 + M_1X_2\hat{\beta}_2 + M_1MY$$

Again the first term of the right hand side vanishes. Now for the third term, MY belongs to the orthogonal complement of $[X_1X_2]$, so further projecting it on the orthogonal complement of X_1 (which is what premultiplying by M_1 would do) has no effect, hence $M_1MY = MY$. This leaves:

$$M_1Y - M_1X_2\hat{\beta}_2 = MY$$

From the first part of the theorem, the left hand side are the errors of projecting $Y*$ on X_2^* and, by definition, the right hand side are the errors of proyecting Y on $[X_1, X_2]$ proving the second part of the theorem.

Figure 1

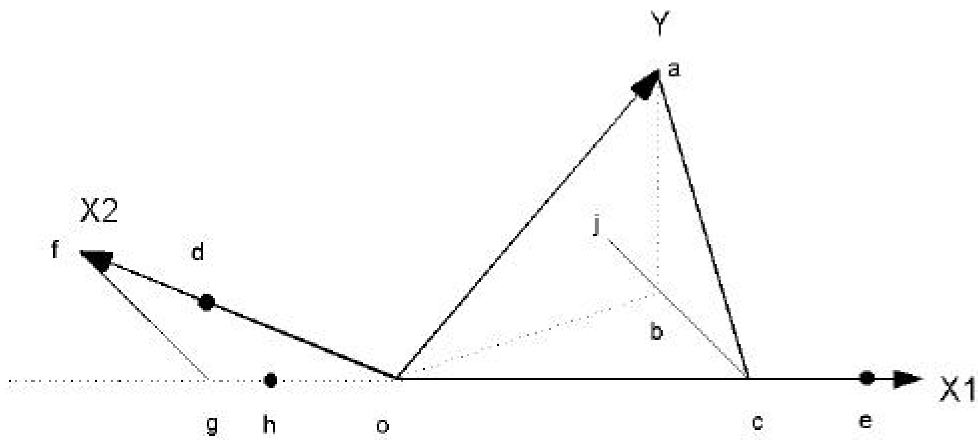

Figure 2